

A MINKOWSKI TYPE INEQUALITY FOR HYPERSURFACES IN THE SCHWARZSCHILD MANIFOLD

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ABSTRACT. In this paper, we use the inverse mean curvature flow to prove a sharp Minkowski type inequality for hypersurfaces in the Schwarzschild manifold.

1. INTRODUCTION

The Schwarzschild manifold is an n -dimensional ($n \geq 3$) manifold $M = [s_0, \infty) \times \mathbb{S}^{n-1}$ equipped with the metric

$$\bar{g} = \frac{1}{1 - 2ms^{2-n}} ds^2 + s^2 g_{\mathbb{S}^{n-1}}, \quad (1)$$

where $m > 0$ is a constant, s_0 is the unique positive solution of $1 - 2ms_0^{2-n} = 0$ and $g_{\mathbb{S}^{n-1}}$ is the canonical round metric on the unit sphere \mathbb{S}^{n-1} . We define the function $f = \sqrt{1 - 2ms^{2-n}}$. The Schwarzschild metric is *asymptotically flat*, that is the sectional curvature of (M, \bar{g}) approach zero near infinity. In fact, by a change of variable, the Schwarzschild manifold (M^n, \bar{g}) can be expressed as $\mathbb{R}^n \setminus \{0\}$ equipped with the conformal metric

$$\bar{g} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} g_{\mathbb{R}^n},$$

where $g_{\mathbb{R}^n}$ is the flat Euclidean metric. Moreover, the scalar curvature of (M, \bar{g}) equals to zero (see §2).

A hypersurface Σ in (M, \bar{g}) is said to be mean convex if its mean curvature H is positive everywhere on Σ . In this paper, we prove the following sharp inequality for mean convex and star-shaped hypersurface in (M, \bar{g}) (see §2 for the definition of star-shaped).

Theorem 1. *Let Σ be a closed mean convex and star-shaped hypersurface in the Schwarzschild manifold (M, \bar{g}) . Then*

$$\int_{\Sigma} f H d\mu \geq (n-1)\omega_{n-1} \left(\left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} - 2m \right), \quad (2)$$

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where ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $|\Sigma|$ is the area of Σ . Moreover, equality holds if and only if Σ is a coordinate sphere $\{s\} \times \mathbb{S}^{n-1}$.

Recall that the boundary $\partial M = \{s_0\} \times \mathbb{S}^{n-1}$ is called the horizon of the Schwarzschild manifold, its area is equal to $|\partial M| = s_0^{n-1} \omega_{n-1}$. Since s_0 is the unique positive solution of $1 - 2ms_0^{2-n} = 0$, we have

$$2m = \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Therefore (2) is equivalent to the following inequality (compare with Theorem 1 in [3])

$$\int_{\Sigma} f H d\mu \geq (n-1) \omega_{n-1}^{\frac{1}{n-1}} \left(|\Sigma|^{\frac{n-2}{n-1}} - |\partial M|^{\frac{n-2}{n-1}} \right). \quad (3)$$

The classical Minkowski inequality for convex hypersurface Σ in \mathbb{R}^n states that

$$\int_{\Sigma} H d\mu \geq (n-1) \omega_{n-1}^{\frac{1}{n-1}} |\Sigma|^{\frac{n-2}{n-1}}. \quad (4)$$

This was generalized by Guan and Li [8] to a mean convex and star-shaped hypersurface using the inverse mean curvature flow. By letting $m \rightarrow 0$, the Schwarzschild metric reduces to the Euclidean metric $\bar{g} = ds^2 + s^2 g_{\mathbb{S}^{n-1}}$ and the potential f becomes $f = 1$. Thus Theorem 1 recover the Minkowski inequality (4) for mean convex and star-shaped hypersurface Σ in \mathbb{R}^n . Note that Huisken's recent work [11] showed that the assumption *star-shaped* in [8] can be replaced by *outward-minimizing*.

Our result is motivated by the recent work of Brendle, Hung and Wang [3], where they proved a sharp Minkowski-type inequality for mean convex and star-shaped hypersurfaces in Anti-deSitter-Schwarzschild manifold (which is *asymptotically hyperbolic* near infinity), by using the inverse mean curvature flow. The inverse mean curvature flow has many applications in geometry and general relativity, see, e.g. [1, 4, 8, 12, 15–17]. We first establish a convergence result for mean convex and star-shaped hypersurface in the Schwarzschild manifold. Note that Huisken and Ilmanen [12] considered the weak solution of the inverse mean curvature flow in asymptotically flat manifold (in a level-set formulation), which includes the Schwarzschild manifold as a special case. In this paper we consider the smooth solution in the notion of Gerhard's work (see [6]). We show that under the inverse mean curvature flow, if the initial hypersurface Σ_0 is mean convex and star-shaped, then the flow hypersurface Σ_t converges to a large coordinate sphere as $t \rightarrow \infty$. Then we define a quantity

$$Q(t) = |\Sigma_t|^{-\frac{n-2}{n-1}} \left(\int_{\Sigma_t} f H d\mu_t + 2(n-1)m\omega_{n-1} \right),$$

and show that $Q(t)$ is monotone decreasing along the inverse mean curvature flow. Thus we could compare the initial value $Q(0)$ with the limit

$\liminf_{t \rightarrow \infty} Q(t)$. Fortunately, we could get a lower bound of this limit

$$\liminf_{t \rightarrow \infty} Q(t) \geq (n-1)\omega_{n-1}^{\frac{1}{n-1}}.$$

From this we can complete the proof of Theorem 1 easily.

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2. PRELIMINARIES

In this section, we collect some facts about the Schwarzschild manifolds and star-shaped hypersurfaces.

2.1. Schwarzschild metric. By a change of variable, the Schwarzschild metric (1) can be written as

$$\bar{g} = dr^2 + \lambda^2(r)g_{\mathbb{S}^{n-1}}, \quad (5)$$

where $\lambda(r)$ satisfies

$$\lambda'(r) = \sqrt{1 - 2m\lambda^{2-n}}.$$

Let $\theta = \{\theta^j\}$, $j = 1, \dots, n-1$ be a coordinate system on \mathbb{S}^{n-1} and ∂_{θ^i} be the corresponding coordinate vector field in M . Let ∂_r be the radial vector. By a direct calculation (see for example [18]), the curvature tensor of (M, \bar{g}) has the following components

$$\begin{aligned} \bar{R}_{ijkl} &= \frac{1 - \lambda'^2}{\lambda^2} (\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}) \\ \bar{R}_{irjr} &= -\frac{\lambda''}{\lambda} \bar{g}_{ij}, \end{aligned}$$

where $1 \leq i, j, k, l \leq n-1$ and $\bar{g}_{ij} = \bar{g}(\partial_{\theta^i}, \partial_{\theta^j}) = \lambda^2 g_{\mathbb{S}^{n-1}}(\partial_{\theta^i}, \partial_{\theta^j}) = \lambda^2 \sigma_{ij}$. Other components of the curvature tensor are equal to zero. Noting that

$$\frac{1 - \lambda'^2}{\lambda^2} = 2m\lambda^{-n}, \quad -\frac{\lambda''}{\lambda} = -m(n-2)\lambda^{-n}. \quad (6)$$

By a further calculation, we have the Ricci curvature of (M, \bar{g})

$$\begin{aligned} \overline{Ric} &= \left((n-2) \frac{1 - \lambda'^2}{\lambda^2} - \frac{\lambda''}{\lambda} \right) \bar{g} - (n-2) \left(\frac{1 - \lambda'^2}{\lambda^2} + \frac{\lambda''}{\lambda} \right) dr^2 \\ &= m(n-2)\lambda^{-n} \bar{g} - mn(n-2)\lambda^{-n} dr^2 \end{aligned} \quad (7)$$

and the scalar curvature

$$\bar{R} = (n-1) \left((n-2) \frac{1 - \lambda'^2}{\lambda^2} - \frac{2\lambda''}{\lambda} \right) = 0. \quad (8)$$

On the other hand, by the definition of f , we have $f = \lambda' = \sqrt{1 - 2m\lambda^{2-n}}$. In the sequel, we denote by $\bar{\nabla}$, $\bar{\nabla}^2$ and $\bar{\Delta}$ the gradient, Hessian and Laplacian

operator on (M, \bar{g}) . We can compute the Hessian of f :

$$\begin{aligned}\bar{\nabla}^2 f &= \frac{\lambda' \lambda''}{\lambda} \bar{g} + (\lambda''' - \frac{\lambda' \lambda''}{\lambda}) dr^2 \\ &= m(n-2) \lambda^{-n} \lambda' \bar{g} - mn(n-2) \lambda^{-n} \lambda' dr^2\end{aligned}\quad (9)$$

Thus we have

$$\bar{\Delta} f = (n-1) \frac{\lambda' \lambda''}{\lambda} + \lambda''' = 0. \quad (10)$$

Combining (7), (9) and (10), we conclude that f satisfies the following static equation:

$$(\bar{\Delta} f) \bar{g} - \bar{\nabla}^2 f + f \bar{Ric} = 0. \quad (11)$$

2.2. Star-shaped hypersurface. We say a hypersurface Σ in (M, \bar{g}) is star-shaped if $\langle \partial_r, \nu \rangle > 0$ on Σ . A star-shaped hypersurface could be parameterized by a graph

$$\Sigma = \{(r(\theta), \theta) : \theta \in \mathbb{S}^{n-1}\}$$

for a smooth function r on \mathbb{S}^{n-1} . As in [3, 5, 7], we define a function φ on \mathbb{S}^{n-1} by

$$\varphi(\theta) = \Phi(r(\theta)),$$

where $\Phi(r)$ is a positive function satisfying $\Phi'(r) = 1/\lambda(r)$. Define

$$v = \sqrt{1 + |D\varphi|_{\mathbb{S}^{n-1}}^2},$$

where D denotes the Levi-Civita connection on \mathbb{S}^{n-1} . Let $\theta = \{\theta^j\}$, $j = 1, \dots, n-1$ be a coordinate system on \mathbb{S}^{n-1} . The unit normal vector of this hypersurface could be written as

$$\nu = \frac{1}{v} (\partial_r - \frac{r^j}{\lambda^2} \partial_{\theta^j}).$$

We can express the metric and second fundamental form of Σ as following (see [3, 5, 7])

$$g_{ij} = \lambda^2 (\sigma_{ij} + \varphi_i \varphi_j), \quad (12)$$

$$h_{ij} = \frac{\lambda'}{v\lambda} g_{ij} - \frac{\lambda}{v} \varphi_{ij}, \quad (13)$$

where φ_i, φ_{ij} are covariant derivatives of φ with respect to the metric $g_{\mathbb{S}^{n-1}}$. After lifting the indice j , we have

$$h_i^j = \frac{1}{v\lambda} \left(\lambda' \delta_i^j - \tilde{\sigma}^{jk} \varphi_{ki} \right), \quad (14)$$

where $\tilde{\sigma}^{jk} = \sigma^{jk} - \frac{\varphi^j \varphi^k}{v^2}$ with $\varphi^j = \sigma^{jk} \varphi_k$. The mean curvature H then has the form

$$H = \frac{(n-1) \lambda' - \tilde{\sigma}^{ij} \varphi_{ij}}{v\lambda}. \quad (15)$$

3. INVERSE MEAN CURVATURE FLOW

In this section, we consider the inverse mean curvature flow in the Schwarzschild manifold, which is a family $X : \Sigma \times [0, T) \rightarrow (M, \bar{g})$ satisfying

$$\partial_t X = \frac{1}{H} \nu, \quad (16)$$

where ν is the unit outward normal and H is the mean curvature of $\Sigma_t = X(\Sigma, t)$. If the initial hypersurface is star-shaped and mean convex, the short time existence result implies the flow exists on a maximum time interval $[0, T)$.

Let $\partial_i, i = 1, 2, \dots, n-1$ be coordinate vector fields on Σ_t . Denote by g_{ij} and h_{ij} the components of the first and second fundamental form, by $H = g^{ij} h_{ij}$ the mean curvature and $|A|^2 = h_{ik} h_{lj} g^{il} g^{jk}$ the squared norm of the second fundamental form, by $\chi = \langle \lambda \partial_r, \nu \rangle$ the support function and by $d\mu_t$ the area element on Σ_t . We first collect the evolution equations for various geometric quantities under the inverse mean curvature flow.

Lemma 2 (Evolution equations). *Under the flow (16), we have*

$$\partial_t g_{ij} = 2H^{-1} h_{ij}, \quad (17)$$

$$\partial_t d\mu_t = d\mu_t, \quad (18)$$

$$\partial_t \nu = \frac{1}{H^2} \nabla H, \quad (19)$$

$$\begin{aligned} \partial_t h_i^j = & \frac{\Delta h_i^j}{H^2} + \frac{|A|^2}{H^2} h_i^j - \frac{2}{H} h_i^k h_k^j - \frac{2}{H^3} \nabla_i H \nabla^j H - \frac{2}{H} \bar{R}_{\nu i \nu k} g^{kj} \\ & + \frac{2}{H^2} g^{lj} g^{ks} h_k^m \bar{R}_{misl} + \frac{1}{H^2} g^{lj} g^{ks} h_i^m \bar{R}_{mksl} + \frac{1}{H^2} g^{lj} g^{ks} h_l^m \bar{R}_{mksi} \\ & + \frac{1}{H^2} \bar{Ric}(\nu, \nu) h_i^j + \frac{1}{H^2} g^{lj} g^{ks} (\bar{\nabla}_k \bar{R}_{\nu isl} + \bar{\nabla}_l \bar{R}_{\nu ksi}) \end{aligned} \quad (20)$$

$$\partial_t H = \frac{\Delta H}{H^2} - 2 \frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\bar{Ric}(\nu, \nu)}{H} \quad (21)$$

$$\partial_t \chi = \frac{1}{H^2} \Delta \chi + \frac{|A|^2}{H^2} \chi - \frac{1}{H^2} \bar{Ric}(\nu, \partial_k) \langle \lambda \partial_r, \partial_j \rangle g^{kj}. \quad (22)$$

where ∇ and Δ are gradient and Laplacian operator with respect to the induced metric on the flow hypersurface Σ_t .

Proof. The evolution equations for the metric, area element, unit normal and the second fundamental form can be calculated in a standard way as in [10], we omit the argument here. For the support function, we have

$$\partial_t \chi = \partial_t \langle \lambda \partial_r, \nu \rangle = \frac{\lambda'}{H} + \frac{1}{H^2} \langle \lambda \partial_r, \nabla H \rangle,$$

where we used the conformal property of the vector field $\lambda \partial_r$ (see [2]) and (19). On the other hand, by using the conformal property of $\lambda \partial_r$ again and the Codazzi equations, we have

$$\nabla_i \chi = \langle \lambda \partial_r, h_i^k \partial_k \rangle$$

and

$$\begin{aligned}\nabla_j \nabla_i \chi &= \lambda' h_{ij} - h_i^k h_{kj} \chi + \langle \lambda \partial_r, \nabla_j h_i^k \partial_k \rangle \\ &= \lambda' h_{ij} - h_i^k h_{kj} \chi + \langle \lambda \partial_r, \nabla^k h_{ij} \partial_k \rangle + \langle \lambda \partial_r, \bar{R}_{\nu ilj} g^{lk} \partial_k \rangle.\end{aligned}$$

Thus we obtain

$$\Delta \chi = \lambda' H - |A|^2 \chi + \langle \lambda \partial_r, \nabla H \rangle + \bar{Ric}(\nu, \partial_l) \langle \lambda \partial_r, \partial_k \rangle g^{kl}.$$

Then (22) follows from combining the above equations. \square

We could use the evolution equation (22) of the support function to show that under the inverse mean curvature flow, the evolved hypersurface Σ_t remains star-shaped.

Lemma 3. *Under the inverse mean curvature flow (16), the evolved hypersurface Σ_t remains star-shaped if Σ_0 is star-shaped.*

Proof. From the expression (7) of Ricci curvature of the Schwarzschild manifold and the evolution equation (22) of the support function, we have

$$\partial_t \chi = \frac{1}{H^2} \Delta \chi + \frac{|A|^2}{H^2} \chi + \frac{1}{H^2} mn(n-2) \lambda^{-n} |\partial_r^T|^2 \chi, \quad (23)$$

where

$$|\partial_r^T|^2 = \sum_{k,j=1}^{n-1} \langle \partial_r, \partial_k \rangle \langle \partial_r, \partial_j \rangle g^{kj}$$

is squared norm of the tangential part ∂_r^T of the radial vector ∂_r . Since $\chi > 0$ on the initial hypersurface Σ_0 , in view of the inequality $|A|^2 \geq H^2/(n-1)$ and using the parabolic maximum principle, we conclude that

$$\chi \geq e^{\frac{t}{n-1}} \min_{\Sigma_0} \chi > 0$$

which implies the star-shapedness of Σ_t . \square

The flow equation (16) is often called the parametric form of the flow. Since each Σ_t is star-shaped, it can also be represented as a graph

$$\Sigma_t = \{(r(\theta, t), \theta) : \theta \in \mathbb{S}^{n-1}\}$$

Then the flow equation is equivalent to the following non-parametric form of the flow (cf. [3, 5, 7])

$$\frac{\partial r}{\partial t} = \frac{v}{H}. \quad (24)$$

The speed function $\frac{v}{H}$ depends on $r, Dr, D^2 r$. It is easy to see that the flow equation (24) is parabolic. In deed, as in section 2 we define $\varphi(\theta, t) = \Phi(r(\theta, t))$ with $\Phi(r)$ is a positive function satisfying $\Phi'(r) = 1/\lambda(r)$. Then

$$\varphi_i = \frac{r_i}{\lambda}, \quad \varphi_{ij} = \frac{r_{ij}}{\lambda} - \frac{\lambda' r_i r_j}{\lambda^2},$$

and we have

$$H = \frac{(n-1)\lambda'}{v\lambda} - \frac{\tilde{\sigma}^{ij}}{v\lambda^2} (r_{ij} - \frac{\lambda' r_i r_j}{\lambda}). \quad (25)$$

Thus

$$\frac{\partial}{\partial r_{ij}}\left(\frac{v}{H}\right) = \frac{\tilde{\sigma}^{ij}}{H^2\lambda^2}$$

which is nonnegative definite and therefore (24) is parabolic.

Lemma 4. *Let r_1, r_2 be constants such that*

$$r_1 < r(\theta) < r_2$$

holds on the initial hypersurface Σ_0 . Then on Σ_t we have the estimate

$$\lambda(r_1)e^{\frac{t}{n-1}} < \lambda(r(\theta, t)) < \lambda(r_2)e^{\frac{t}{n-1}}, \quad \forall t \in [0, T].$$

Proof. Let $S_{r_i}, i = 1, 2$ be coordinate spheres $\{r_i\} \times \mathbb{S}^{n-1}$. We solve the inverse mean curvature flows with initial hypersurface S_{r_i} respectively. If the initial hypersurface is a coordinate sphere, the inverse mean curvature flow becomes a scalar flow:

$$\frac{dr}{dt} = \frac{1}{H} = \frac{\lambda}{(n-1)\lambda'},$$

where we used that the principal curvatures of a coordinate sphere are λ'/λ . Then

$$\frac{d\lambda}{dt} = \frac{\lambda}{n-1}.$$

From this we deduce that $\lambda(r_i(t)) = \lambda(r_i(0))e^{\frac{t}{n-1}}$. By the parabolic maximum principle, we have

$$r_i(t) < r(\theta, t) < r_2(t), \quad t \in [0, T].$$

Since $\lambda' > 0$, we also have

$$\lambda(r_1(t)) < \lambda(r(\theta, t)) < \lambda(r_2(t)).$$

The assertion follows from the above inequality. \square

Lemma 5. *There is a constant $C_1 > 0$ such that $He^{\frac{t}{n-1}} \leq C_1$.*

Proof. From the evolution equation (21) of H , we have

$$\partial_t H^2 = -2H\Delta\frac{1}{H} - 2|A|^2 - 2\overline{Ric}(\nu, \nu).$$

Using $\overline{Ric}(\nu, \nu) = O(\lambda^{-n}) = O(e^{-\frac{nt}{n-1}})$ and the inequality

$$|A|^2 \geq \frac{1}{n-1}H^2,$$

we obtain that

$$\frac{d}{dt}H_{\max}^2 \leq -\frac{2}{n-1}H_{\max}^2 + O(e^{-\frac{nt}{n-1}}).$$

From this, the assertion follows easily. \square

To estimate the lower bound of H , by the definition of φ we have

$$\frac{\partial \varphi}{\partial t} = \frac{v}{\lambda H} = \frac{1}{F}, \quad (26)$$

where the function

$$F = \frac{\lambda H}{v} = \frac{(n-1)\lambda' - \tilde{\sigma}^{ij}\varphi_{ij}}{v^2}.$$

The flow equation (26) is also parabolic.

Lemma 6. *There is a constant $C_2 > 0$ such that $He^{\frac{t}{n-1}} \geq C_2$.*

Proof. If we differentiate (26) with respect to t , we obtain

$$\partial_t(\partial_t \varphi) = \frac{\tilde{\sigma}^{ij}}{v^2 F^2} (\partial_t \varphi)_{ij} - \frac{1}{F^2} \frac{\partial F}{\partial \varphi_i} (\partial_t \varphi)_i - \frac{(n-1)\lambda\lambda''}{\lambda^2 H^2} \partial_t \varphi.$$

So from (6) we have

$$\frac{d}{dt}(\partial_t \varphi)_{\max} \leq 0.$$

Noting that $\partial_t \varphi = \frac{v}{\lambda H}$ and $v \geq 1$, we obtain that

$$\lambda H \geq C > 0,$$

The assertion follows from the above inequality and by using Lemma 4. \square

For the first space derivatives of φ , we have the following estimate.

Lemma 7. *There is a constant $\beta > 0$ such that $|D\varphi|_{\mathbb{S}^{n-1}} \leq O(e^{-\beta t})$.*

Proof. Let $\omega = \frac{1}{2}|D\varphi|_{\mathbb{S}^{n-1}}^2$. We can compute as lemma 8 in [3] to obtain the evolution of ω :

$$\begin{aligned} \partial_t \omega &= \frac{\tilde{\sigma}^{ij}}{v^2 F^2} \omega_{ij} - \frac{1}{F^2} \frac{\partial F}{\partial \varphi_i} \omega_i - \frac{\tilde{\sigma}^{ij}}{v^2 F^2} \sigma^{kl} \varphi_{ik} \varphi_{jl} \\ &\quad - \frac{2(n-2)}{\lambda^2 H^2} \omega - \frac{2(n-1)\lambda''}{\lambda H^2} \omega. \end{aligned}$$

By Lemma 4 and Lemma 5, there is a constant $\beta > 0$ such that

$$\frac{(n-2)}{\lambda^2 H^2} \geq \beta.$$

So from (6) we have

$$\frac{d}{dt} \omega_{\max} \leq -2\beta \omega_{\max},$$

which implies the Lemma. \square

Next, we will estimate the second fundamental form of Σ_t . We define the tensor M by (see [3, 13])

$$M_i^j = H h_i^j.$$

Combining the evolution equations (20),(21), and using the fact that the Schwarzschild metric is asymptotically flat (see §2) and lemma 4, we have

$$\begin{aligned} \partial_t M_i^j &= \frac{1}{H^2} \Delta M_i^j - \frac{2}{H^3} \nabla^k H \nabla_k M_i^j - \frac{2}{H^2} \nabla^j H \nabla_i H \\ &\quad - \frac{2}{H^2} M_i^k M_k^j + \frac{|M|}{H^2} O(e^{-\frac{nt}{n-1}}) + o(e^{-\frac{n+1}{n-1}t}). \end{aligned} \quad (27)$$

If on the time interval considered we have an uniform bound $H_0 \leq H \leq H_1$ for the mean curvature, by Hamilton's maximum principle for parabolic system [9], we can bound the largest eigenvalue μ_{n-1} of M above by the solution of the following ODE

$$\frac{d}{dt} \varphi = -\frac{2}{H_1^2} \varphi^2 + \frac{\varphi}{H_0^2} O(e^{-\frac{nt}{n-1}}) + o(e^{-\frac{n+1}{n-1}t}).$$

So that μ_{n-1} and then the largest principal curvature κ_{n-1} of Σ_t have a uniform upper bound depending on H_1 and H_0 . Since the mean curvature is bounded from below, it follows that the full second fundamental form is bounded by

$$|A| \leq C(n, H_0, H_1). \quad (28)$$

In view of the mean curvature estimate in Lemma 5 and Lemma 6, we have the long time existence of the inverse mean curvature flow.

Proposition 8. *The solution of the inverse mean curvature flow is defined on $[0, \infty)$.*

Proof. Let $[0, T)$ be the maximum time interval of existence for the smooth solution of the inverse mean curvature flow. If $T < \infty$, then Lemma 5 and Lemma 6 imply that

$$C_2 e^{-\frac{T}{n-1}} \leq H \leq C_1.$$

From (28), we know that the second fundamental form of Σ_t is bounded for $t \rightarrow T$. Then the regularity results of Krylov [14] and the short time existence theorem imply that we can extend the solution smoothly beyond T , contradicting with the maximum of T . So we conclude that T must be ∞ . \square

Finally we show that the solution Σ_t converges to a large coordinate sphere as $t \rightarrow \infty$. Let us define

$$\tilde{\lambda} = \lambda e^{-\frac{t}{n-1}}. \quad (29)$$

Then $\tilde{\lambda}$ satisfies

$$\partial_t \tilde{\lambda} = \frac{\lambda' v}{H} e^{-\frac{t}{n-1}} - \frac{1}{n-1} \tilde{\lambda} \quad (:= \tilde{F}), \quad (30)$$

where we denote the right hand side of (30) by \tilde{F} . By lemma 4, the family $\tilde{\lambda}(\cdot, t)$ is uniformly bounded. By lemma 5 and lemma 6, $|\partial_t \tilde{\lambda}|$ is also uniformly bounded. Noting that

$$\tilde{\lambda}_{ij} = (\lambda'' \lambda^2 + \lambda \lambda'^2) e^{-\frac{t}{n-1}} \varphi_i \varphi_j + \lambda \lambda' e^{-\frac{t}{n-1}} \varphi_{ij} \quad (31)$$

and the expression (15) of H , we deduce that

$$\frac{\partial \tilde{F}}{\partial \tilde{\lambda}_{ij}} = \frac{\tilde{\sigma}^{ij}}{H^2 \lambda^2}$$

which is positive definite and therefore (30) is parabolic. Moreover, from lemma 5, lemma 6 and lemma 7, we conclude that (30) is uniformly parabolic.

By lemma 7, $D\tilde{\lambda}$ decays exponentially fast:

$$D\tilde{\lambda} = D\lambda e^{-\frac{t}{n-1}} = \lambda' \lambda e^{-\frac{t}{n-1}} D\varphi = O(e^{-\beta t}). \quad (32)$$

Thus $\tilde{\lambda}$ converges to a positive constant $\bar{\lambda}$ uniformly. From the regularity estimate of Krylov [14, §5.5], the second derivatives of $\tilde{\lambda}$ are uniformly bounded in $C^{0,\alpha}$. Using the interpolation theorem we deduce that $D^2\tilde{\lambda}$ also decays exponentially fast. In view of (31) and lemma 4, lemma 7, we have $|D^2\varphi| = O(e^{-\tilde{\beta}t})$ for some constant $\tilde{\beta} > 0$.

By (12) and lemma 7, the metric of Σ_t satisfies

$$e^{-\frac{2t}{n-1}} g_{ij} \rightarrow \bar{\lambda}^2 \sigma_{ij} \quad (33)$$

exponentially fast. From the expression (14) of h_i^j , we have

$$\left| \frac{\lambda}{\lambda'} h_i^j - \delta_i^j \right| = \left(\frac{1}{v} - 1 \right) \delta_i^j - \frac{1}{v\lambda'} \tilde{\sigma}^{jk} \varphi_{ki} = O(e^{-\beta' t}) \quad (34)$$

for a positive constant $\beta' = \min\{2\beta, \tilde{\beta}\}$. This implies that Σ_t converges to a large coordinate sphere as $t \rightarrow \infty$.

4. PROOF OF THEOREM 1

In this section, we follow a similar argument in [3] to prove Theorem 1. We define the quantity

$$Q(t) = |\Sigma_t|^{-\frac{n-2}{n-1}} \left(\int_{\Sigma_t} f H d\mu_t + 2(n-1)m\omega_{n-1} \right),$$

where $|\Sigma_t|$ is the area of Σ_t . We first show that $Q(t)$ is monotone decreasing under the inverse mean curvature flow.

Proposition 9. *Under the inverse mean curvature flow (16), the quantity $Q(t)$ is monotone decreasing in t .*

Proof. As the proof of Proposition 19 in [3], we first have

$$\frac{d}{dt} \int_{\Sigma_t} f H d\mu_t \leq \int_{\Sigma_t} \left(\frac{n-2}{n-1} f H + 2 \langle \bar{\nabla} f, \nu \rangle \right) d\mu_t, \quad (35)$$

and equality holds if and only if Σ_t is totally umbilical. For convenience of reader, we include the proof of (35) here.

$$\begin{aligned}
\frac{d}{dt} \int_{\Sigma_t} f H d\mu_t &= \int_{\Sigma_t} (\partial_t f H + f \partial_t H + f H) d\mu_t \\
&= \int_{\Sigma_t} \left(\langle \bar{\nabla} f, \nu \rangle - f \Delta \frac{1}{H} - \frac{f}{H} |A|^2 - \frac{f}{H} \overline{Ric}(\nu, \nu) + f H \right) d\mu_t \\
&\leq \int_{\Sigma_t} \left(\langle \bar{\nabla} f, \nu \rangle - \frac{1}{H} (\Delta f + f \overline{Ric}(\nu, \nu)) + \frac{n-2}{n-1} f H \right) d\mu_t,
\end{aligned} \tag{36}$$

where we used $|A|^2 \geq H^2/(n-1)$ in the last inequality. Using the identity $\Delta f = \bar{\Delta} f - \bar{\nabla}^2 f(\nu, \nu) - H \langle \bar{\nabla} f, \nu \rangle$ and (10),(11), we have

$$\Delta f + f \overline{Ric}(\nu, \nu) = -H \langle \bar{\nabla} f, \nu \rangle.$$

Substituting this into (36) gives (35). If equality holds in (35), then $|A|^2 = H^2/n-1$ and Σ_t is totally umbilical.

Let Ω_t denote the region bounded by Σ_t and the horizon ∂M . Using the divergence theorem and noting that $\bar{\Delta} f = 0$, we get

$$\begin{aligned}
\int_{\Sigma_t} \langle \bar{\nabla} f, \nu \rangle d\mu_t &= \int_{\Omega_t} \bar{\Delta} f d\text{vol} + m(n-2)\omega_{n-1} \\
&= m(n-2)\omega_{n-1}
\end{aligned}$$

which is a constant. Thus we obtain

$$\frac{d}{dt} \left(\int_{\Sigma_t} f H d\mu_t + 2(n-1)m\omega_{n-1} \right) \leq \frac{n-2}{n-1} \left(\int_{\Sigma_t} f H d\mu_t + 2(n-1)m\omega_{n-1} \right).$$

On the other hand, from the evolution (18) of the area element $d\mu_t$, the area of $|\Sigma_t|$ satisfies $\frac{d}{dt} |\Sigma_t| = |\Sigma_t|$. So we conclude that

$$\frac{d}{dt} Q(t) \leq 0,$$

equality holds if and only if (35) assumes equality and then Σ_t is totally umbilical. \square

Once we obtain the monotonicity of $Q(t)$, we need to investigate the limit of $Q(t)$ as $t \rightarrow \infty$.

Proposition 10. *We have*

$$\liminf_{t \rightarrow \infty} Q(t) \geq (n-1)\omega_{\frac{n-1}{n-1}}.$$

Proof. From the expression (12) of the metric g and lemma 7, the area element $d\mu_t$ satisfies

$$d\mu_t = \lambda^{n-1} (1 + O(e^{-2\beta t})) d\text{vol}_{\mathbb{S}^{n-1}},$$

Then we have

$$|\Sigma_t|^{\frac{n-2}{n-1}} = \left(\int_{\mathbb{S}^{n-1}} \lambda^{n-1} d\text{vol}_{\mathbb{S}^{n-1}} \right)^{\frac{n-2}{n-1}} (1 + O(e^{-2\beta t})). \quad (37)$$

On the other hand,

$$f = \sqrt{1 - 2m\lambda^{2-n}} = 1 - m\lambda^{2-n} + O(\lambda^{4-2n}).$$

By the expression (15) of the mean curvature H and the exponentially decay of φ_i, φ_{ij} ,

$$\begin{aligned} \lambda H &= \frac{(n-1)\lambda'}{v} - \frac{\sigma^{ij}\varphi_{ij}}{v} + \frac{\varphi^i\varphi^j\varphi_{ij}}{v^3} \\ &= n-1 + O(e^{-\gamma t}) \end{aligned}$$

for some positive constant $\gamma = \min\{2\beta, \tilde{\beta}, \frac{n-2}{n-1}\}$. So we have

$$\int_{\Sigma_t} f H d\mu_t = (n-1) \int_{\mathbb{S}^{n-1}} \lambda^{n-2} d\text{vol}_{\mathbb{S}^{n-1}} (1 + O(e^{-\gamma t})). \quad (38)$$

Then we obtain

$$\liminf_{t \rightarrow \infty} Q(t) \geq (n-1) \liminf_{t \rightarrow \infty} \frac{\int_{\mathbb{S}^{n-1}} \lambda^{n-2} d\text{vol}_{\mathbb{S}^{n-1}}}{\left(\int_{\mathbb{S}^{n-1}} \lambda^{n-1} d\text{vol}_{\mathbb{S}^{n-1}} \right)^{\frac{n-2}{n-1}}} \quad (39)$$

Since $\tilde{\lambda} = \lambda e^{-\frac{t}{n-1}}$ converges to a constant $\bar{\lambda}$, there exists a positive function $\epsilon(t)$ such that $\lim_{t \rightarrow \infty} \epsilon(t) = 0$ and

$$\bar{\lambda} - \epsilon < \tilde{\lambda} < \bar{\lambda} + \epsilon$$

or equivalently

$$(\bar{\lambda} - \epsilon)e^{\frac{t}{n-1}} < \lambda < (\bar{\lambda} + \epsilon)e^{\frac{t}{n-1}}$$

when t is sufficiently large. So we conclude that

$$\liminf_{t \rightarrow \infty} Q(t) \geq (n-1) \omega_{n-1}^{\frac{1}{n-1}} \liminf_{t \rightarrow \infty} \left(\frac{\bar{\lambda} - \epsilon}{\bar{\lambda} + \epsilon} \right)^{n-2} = (n-1) \omega_{n-1}^{\frac{1}{n-1}}, \quad (40)$$

which completes the proof. \square

Proof of Theorem 1. Since $Q(t)$ is monotone decreasing in t , we have

$$Q(0) \geq \liminf_{t \rightarrow \infty} Q(t) \geq (n-1) \omega_{n-1}^{\frac{1}{n-1}}.$$

Thus we obtain

$$\int_{\Sigma_t} f H d\mu_t + 2(n-1)m\omega_{n-1} \geq (n-1) \omega_{n-1}^{\frac{1}{n-1}} |\Sigma_t|^{\frac{n-2}{n-1}}$$

which is equivalent to (2). If the equality holds in (2), then $Q(t)$ is a constant in t . From the proof of Proposition 9, the hypersurface Σ_0 is totally umbilical. It follows from the Codazzi equations that $\overline{\text{Ric}}(\nu, e_i) = 0$ for any tangent vector fields e_i . Since $m > 0$, the expression (7) of Ricci curvature implies that the radial vector ∂_r is either parallel or orthogonal to the unit

normal vector ν of Σ_0 . By the star-shapedness of Σ_0 , ∂_r is parallel to ν at each point of Σ_0 and then Σ_0 is a coordinate sphere $\{s\} \times \mathbb{S}^{n-1}$. \square

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